Physics 618 2020

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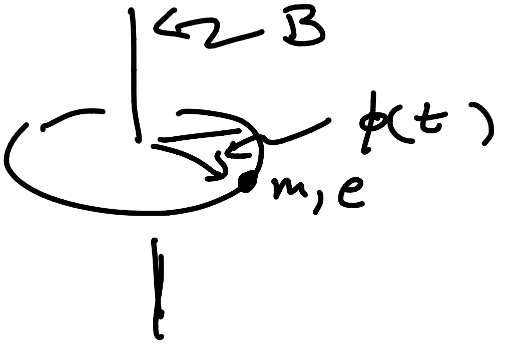
Adding a Potential to Particle On a Ring



- $$S = \int \frac{1}{2} I \dot{\phi}^2 dt + \int \frac{eB}{2\pi} \dot{\phi}\, dt$$

- $$H_{\mathcal{B}} = \frac{\hbar^2}{2I}\left(-i\frac{\partial}{\partial\phi} - \mathcal{B}\right)^2 \quad \text{on} \quad \underline{L^2(S^1)}$$

- Classical $O(2) = \langle P, R(\alpha)\rangle$
  - $P: \phi \to -\phi$
  - $R(\alpha): \phi \to \phi + \alpha$

$$\boxed{P R(\alpha) P = R(\alpha)^{-1} = R(-\alpha)} \quad O(2) = SO(2) \rtimes \mathbb{Z}_2$$

Quantum: $2\mathcal{B} \in \mathbb{Z}$ (parity symmetry)

$$\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad \text{e.v. of } H_{\mathcal{B}}$$

$\rightarrow$ $\mathcal{R}(\alpha): \quad \Psi_m \longrightarrow e^{im\alpha}\Psi_m$

$\mathcal{P}: \quad \Psi_m \longrightarrow \Psi_{2\mathcal{B}-m}$

$$\mathcal{P}\,\mathcal{R}(\alpha)\,\mathcal{P} = e^{i2\mathcal{B}\alpha}\,\mathcal{R}(-\alpha)$$

not w.d. when $\mathcal{B}$ ½ ints

$\alpha \sim \alpha + 2\pi$

$2\mathcal{B} \in 2\mathbb{Z}$ modify $\mathcal{R}(\alpha)$ by

Coboundary $\boxed{\tilde{\mathcal{R}}(\alpha) = e^{-i\mathcal{B}\alpha}\,\mathcal{R}(\alpha)}$

$$\mathcal{P}\,\tilde{\mathcal{R}}(\alpha)\,\mathcal{P} = \tilde{\mathcal{R}}(\alpha)^{-1} = \tilde{\mathcal{R}}(-\alpha)$$

realizes $O(2)$ on Hilbert space

<u>But</u>, if $2\mathcal{B}$ odd integer

$$Spin(2) \xrightarrow{\pi} SO(2)$$

$$\underline{\underline{\hat{R}(\hat{\alpha})}} = \exp(\hat{\alpha}\sigma^1\sigma^2)$$

$$\longrightarrow \underline{\underline{R(2\hat{\alpha})}}$$

Extend to

$$\mathrm{Pin}^+(2) \xrightarrow{\overset{2:1}{\pi}} O(2)$$

$$\langle \hat{P}, \hat{R}(\hat{\alpha}) \rangle \longrightarrow \langle P, R(\alpha) \rangle$$

$$\hat{P}\,\hat{R}(\hat{\alpha})\,\hat{P} = \hat{R}(\hat{\alpha})^{-1}$$

We can represent $\mathrm{Pin}^+(2)$ on $\mathcal{H}$

$$\left\{ \begin{array}{l} \boxed{\rho(\hat{R}(\underline{\hat{\alpha}})) = e^{-i(2\mathcal{B})\hat{\alpha}}\,\mathcal{R}(\underline{\underline{2\hat{\alpha}}})} \\ \rho(\hat{P}) = \mathcal{P} \end{array} \right.$$

Satisfy defining rel's of $\mathrm{Pin}^+(2)$.

e.g. $\mathcal{B} = 1/2$ $\quad \mathcal{H}_{gnd} = \mathrm{Span}\{\underline{\underline{\psi_0, \psi_1}}\}$

$$\rho(\hat{R}(\hat{\alpha}))\Big|_{\mathcal{H}_{gnd}} = \begin{pmatrix} e^{-i\hat{\alpha}} & \\ & e^{i\hat{\alpha}} \end{pmatrix}$$

$$\rho(\hat{P})\Big|_{\mathcal{H}_{gnd}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\boxed{H_{\mathcal{B}} + U(\phi)}$ where $\boxed{2\mathcal{B} = \text{odd}}$

$$U(\phi) = \sum \underline{u_n} \cos(2n\phi)$$

breaks $O(2) \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\mathbb{Z}_2 = \langle P \rangle, \quad \mathbb{Z}_2 = \langle r \rangle$$

$$r : \phi \longrightarrow \phi + \pi$$

motivates looking @ D_4

$$\begin{tikzcd} 1 \arrow[r] & \underline{\mathbb{Z}_2} \arrow[r, "i"] \arrow[d, equal] & \underline{D_4} \arrow[r, "\pi"] \arrow[d] & \underline{\mathbb{Z}_2 \times \mathbb{Z}_2} = \langle P, r \rangle \arrow[d] \\ 1 \arrow[r] & \mathbb{Z}_2 \arrow[r] & Pin^+(2) \arrow[r] & O(2) \arrow[r] & 1 \end{tikzcd}$$

$$\pm \hat{r} \xleftarrow{s} r$$

$$\underline{\hat{r}} = e^{i\mathcal{B}\pi} \mathcal{R}(\pi)$$

$\mathcal{R}, \mathcal{P}$ operate on $\mathcal{H} = L^2(S^1)$

$$\pm \underline{\mathcal{P}} \xleftarrow{s} P$$

these operators commute w/ H @ $u_n = 0$!!

$\langle P, \hat{r} \rangle$ generate a group of operators on $\mathcal{H} \cong D_4$

$$\boxed{D_4 : \langle x, y \mid x^2 = 1,\ y^4 = 1 \quad xyx = y^{-1} \rangle}$$

(Symmetries of the square)

$$P^2 = 1 \qquad \hat{r}^4 = 1 \qquad P \hat{r} P = \hat{r}^{-1}$$

In addition $\hat{r}^2 = -\mathbb{1}$ on $\mathcal{H}$ is not a relation of D_4, but it is compatible and it tells us what rep's appear.

$$y \longleftrightarrow \hat{r}, \quad \hat{r}^2 = -1$$

@ $u_n = 0$ ↓

$$\boxed{\mathcal{H} = \bigoplus \text{ 2-dim'l eigenspaces} \atop \text{all 2d irreps of } D_4}$$

Representations of D_4

1-dimensional representations

In a 1-dim'l rep.

$$\rho(x) = X \in \mathbb{C}$$
$$\rho(y) = \xi \in \mathbb{C}$$

Satisfy relations defining a group

$$X^2 = 1 \qquad \xi^4 = 1 \qquad \underbrace{X \xi X = \xi^{-1}}_{\xi = \xi^{-1} \Rightarrow \underline{\xi^2 = +1}}$$

$$\underline{X^2 = 1}\,, \quad \underline{\xi^2 = 1}$$

Conclusion: There are <u>four</u> distinct 1-dim'l irred. reps of D_4

$$X = \pm 1 \quad \text{and} \quad \xi = \pm 1$$

Fact about reps of finite groups

Finite # of distinct irreps

$$d_\mu \qquad \mu = 1, \dots, s$$

$$|G| = \sum_\mu d_\mu^2$$

(Simple corallary of Peter-Weyl thm.)

$$8 = 4 \cdot 1^2 + 2^2$$

2 diml irrep : Action on the Qbit ground state

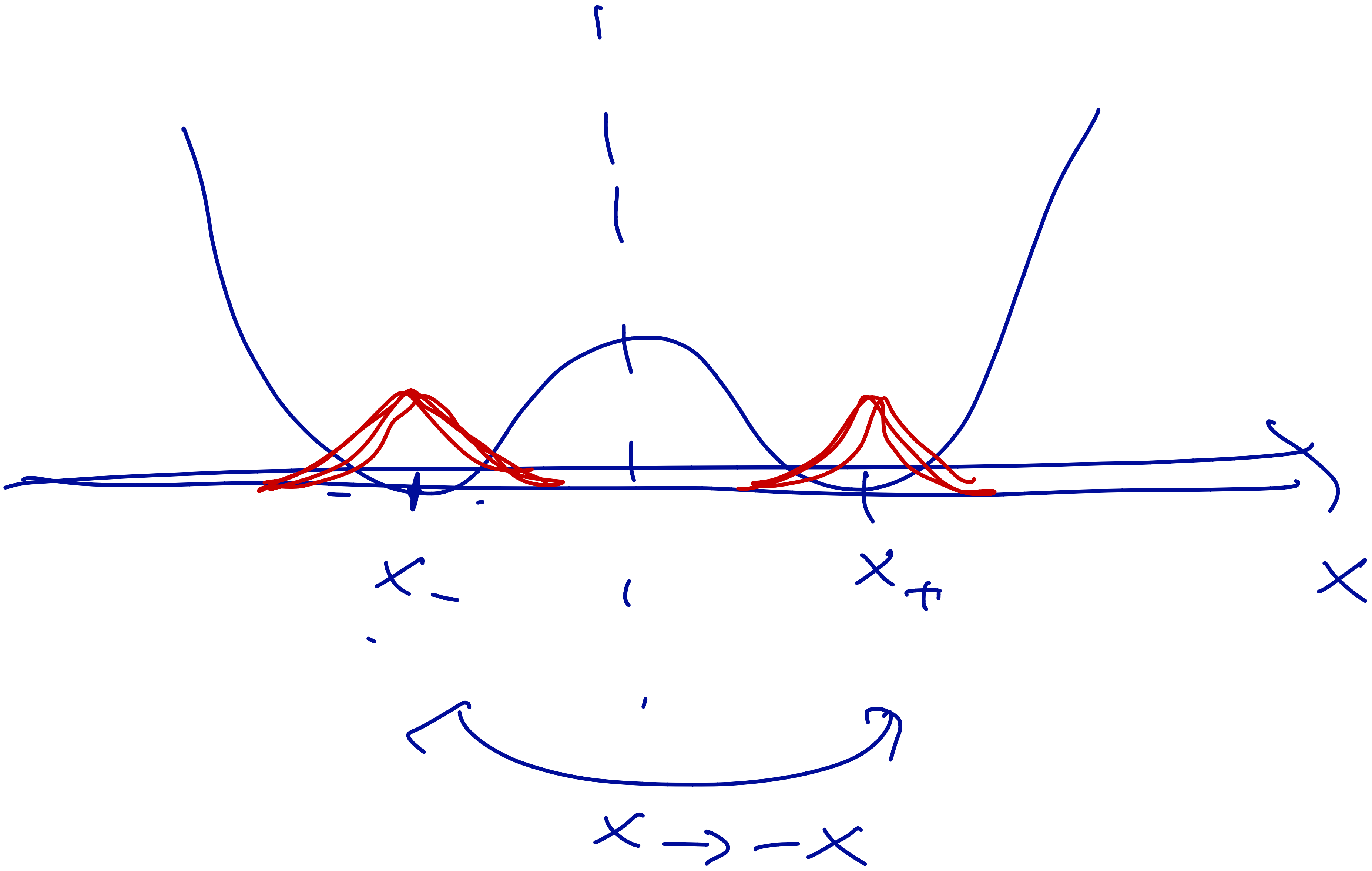
$$\rho(y) = \begin{pmatrix} i & \\ & -i \end{pmatrix}$$

$$\rho(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Irreducible. If it were reducible

$$S \begin{pmatrix} i & \\ & -i \end{pmatrix} S^{-1} = \begin{pmatrix} \xi_1 & \\ & \xi_2 \end{pmatrix} \begin{matrix} \longleftarrow \\ \longleftarrow \end{matrix} \text{ 1-diml reps.}$$

$$S \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix} \begin{matrix} \longleftarrow \\ \longleftarrow \end{matrix}$$



Perturbatively two-fold degeneracy

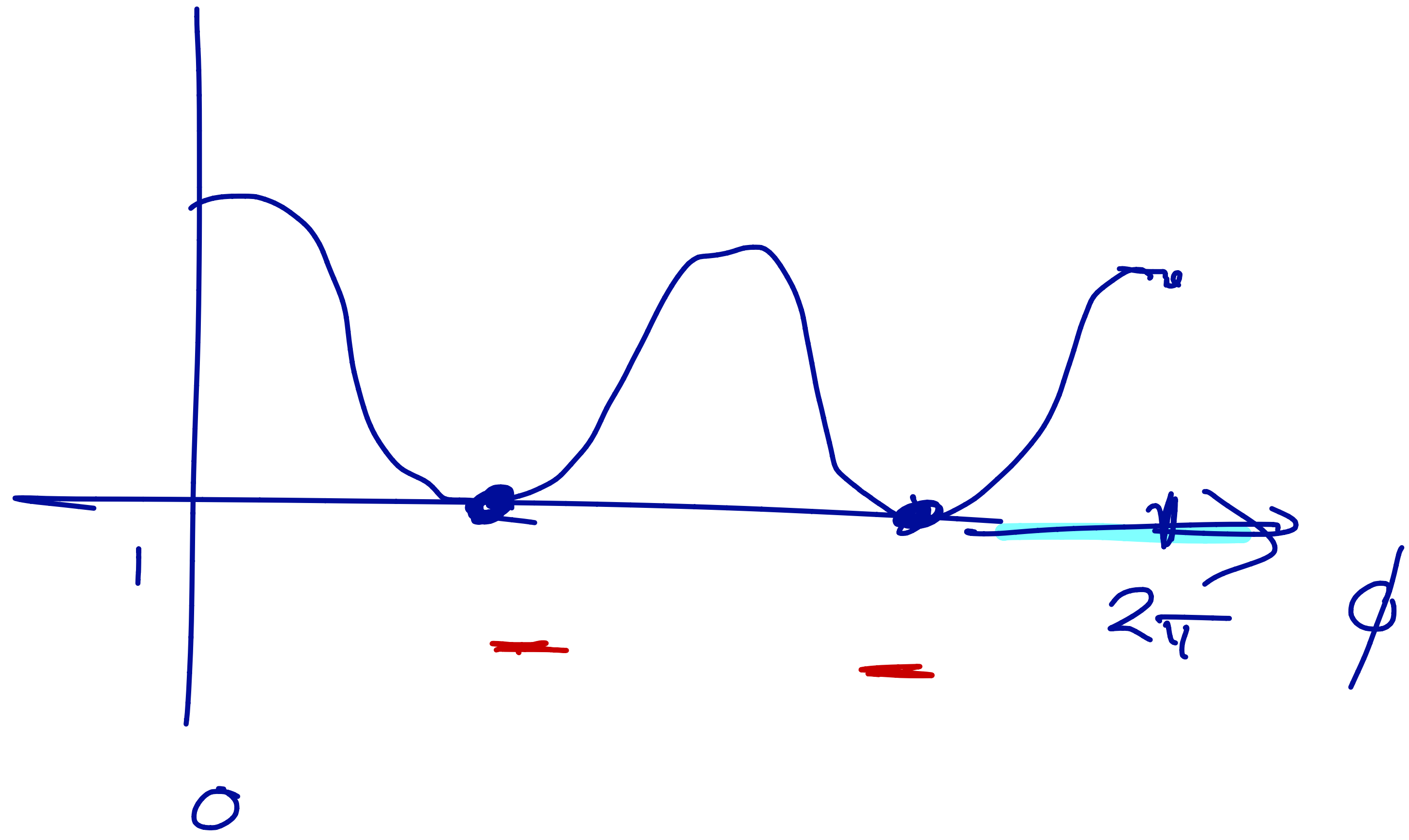
nonperturbatively these mix and we get a 1-dim'l groundstate.

No such S' exists.

Ground state @ $u_n \neq 0$ is 2 dim'l irrep of D_4 !!

(Again assuming decomposition into irreps is a continuous function of u_n.)

Remarkable:



In double-well potential

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This group theory shows that 2B odd groundstate degen. persists and tunneling effects do not spoil it!

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Quantum Statistical Mechanics of this system:

$$Z := \mathrm{Tr}_{\mathcal{H}}\left(e^{-\beta H}\right)$$ partition function.

$$\beta = \frac{1}{kT}$$

T = abs. temp

k = Boltzmann cnst.

$k = \hbar = 1$. henceforth.

In our example $\mathcal{H} = L^2(S^1)$

$H = H_{B}$

$$\boxed{Z = \sum_{m\in\mathbb{Z}} e^{-\frac{\beta}{2I}(m-\mathcal{B})^2}}$$

$\uparrow$

ψ_m

function of $\frac{\beta}{2I}$ and $\mathcal{B}$.

Note: Manifestly periodic in $\mathcal{B}$

$\mathcal{B} \to \mathcal{B}+1$ undo that $m \to m+1$

$$U H_{\mathcal{B}} U^{-1} = H_{\mathcal{B}+1}$$

$e^{-\beta E}/Z$

$\boxed{\beta\to\infty}$

$\beta \longrightarrow 0, \infty$ interesting limits

$\beta \to \infty$, $T \searrow 0$, dominant terms come from groundst.

$\beta \to 0$, $T \nearrow \infty$, "all terms contribute about the same"

expect a divergence.

Underneath that divergence is a very interesting duality.

Evaluate Z in a different way:
Using a path integral.

$$Z = \int_0^{2\pi} d\phi \; \underbrace{\langle \phi | e^{-\beta H_B} | \phi \rangle}_{\text{Special case of}}$$

position eigenstates

$$\langle \phi_2 | e^{-t_E H_B} | \phi_1 \rangle \quad @ \quad \begin{array}{l} \phi_2 = \phi_1 \\ t_E = \beta \end{array}$$

In solving Schrödinger eq.

$$i\hbar \frac{\partial \psi}{\partial t} = H \psi$$

$$\psi(t) = U(t) \, \psi(0)$$

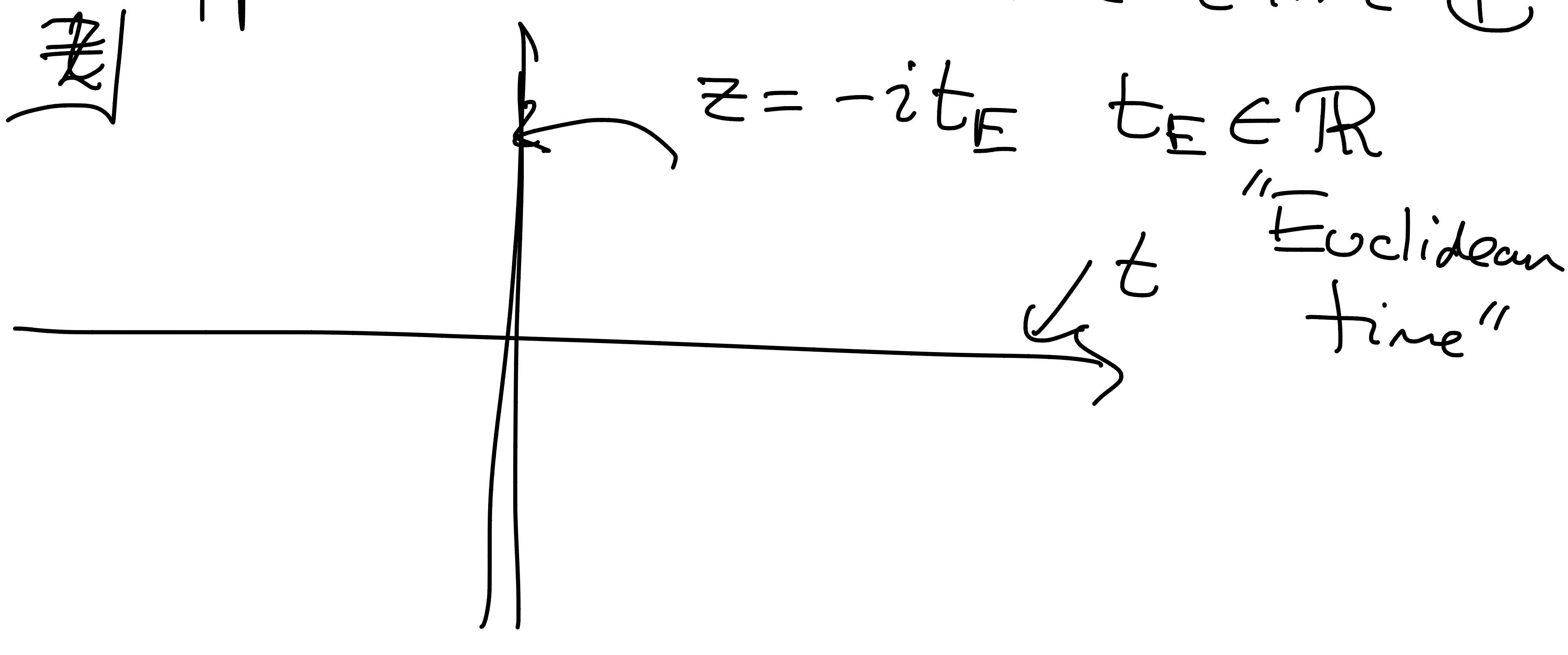
$$U(t) = e^{-i\frac{t}{\hbar}H}$$

Under good conditions this admits an "analytic continuation" to a region in the complex t-plane

$$U(z) = e^{-i\frac{z}{\hbar}H} \qquad z \in \underline{\underline{\mathcal{R}}} \subset \underline{\underline{\mathbb{C}}}$$

so that
$$\begin{array}{ccc}\mathbb{R} & \subset & \mathbb{C}\\ \cap & & \\ \overline{\mathcal{R}} & & \end{array}$$

Suppose $\mathcal{R}$ were the entire $\mathbb{C}$



$z = -it_E \quad t_E \in \mathbb{R}$

"Euclidean time"

$$-(dt)^2 + (dx^i)^2 \qquad t \to -it_E$$

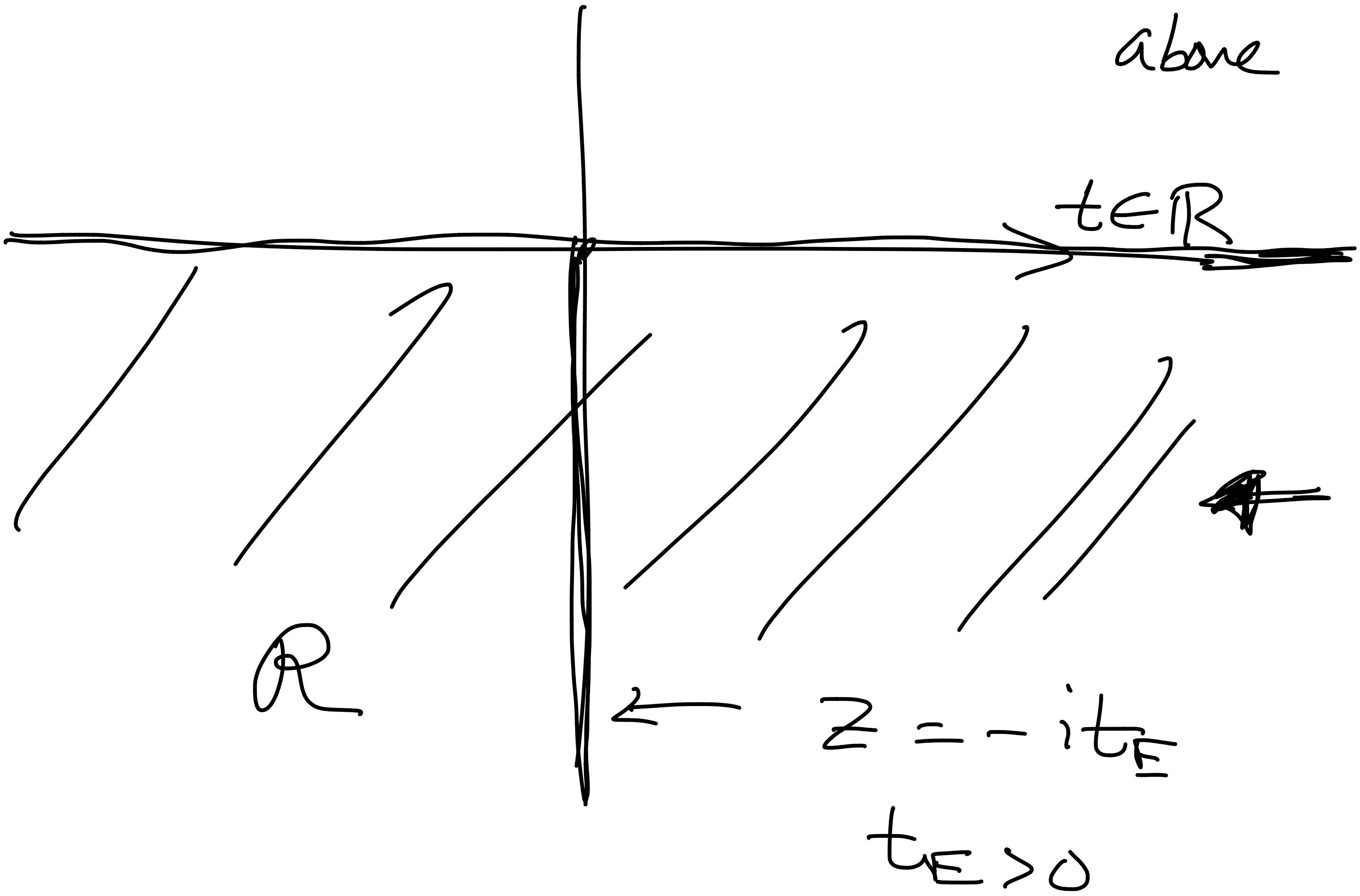
$$(dt_E)^2 + (dx^i)^2$$

$\hbar = 1$

$e^{-t_E H}$

for $t_E > 0$ H is bounded below this is a "good" (traceclass) operator.

for $t_E < 0$ H unbound above



"Analytic continuation to Euclidean time"

"Wick rotation"

In QM matrix element for propagation

$$\langle \phi_2 | e^{-itH} | \phi_1 \rangle$$

has a path integral interpretation.

$$= \int [d\phi(t')]_{\phi(0)=\phi_1}^{\phi(t)=\phi_2} e^{\frac{i S[\phi(t')]}{\hbar}}$$

This path integral realization has a Wick rotation.

$$\boxed{\begin{aligned}&\langle \phi_2 | e^{-\beta H} | \phi_1 \rangle := Z(\phi_2, \phi_1 | \beta)\\ &= \int [d\phi(t)]_{\phi(0)=\phi_1}^{\phi(\beta)=\phi_2} \exp\left[-\frac{1}{\hbar} \int_0^\beta \frac{1}{2} I \dot{\phi}^2 dt - i \int_0^\beta \mathcal{B} \dot{\phi}\, dt \right]\end{aligned}}$$

Euclidean time, dropped the subscript "E"

as a field theory this is free field theory Gaussian action $\Longrightarrow$ evaluate exactly.

$$Z = \int_0^{2\pi} d\phi \, \langle \phi | e^{-\beta H} | \phi \rangle$$

$$\phi_2 = \phi(\beta) = \phi_1 = \phi(0)$$

$$\phi \sim \phi + 2\pi$$

$$\phi(t) : [0, \beta] \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$$

$\phi(0) = \phi(\beta)$ $\phi(t)$ is defined on S^1 = Circle of Euclidean time radius β.

$$\begin{tikzcd}
e^{i\phi(t)} : M \arrow[r] \arrow[d, phantom, sloped, "="] & X \arrow[dd, phantom, sloped, "="] \arrow[dd, phantom, "\text{this case}"'] & \text{target space} \\
\text{Euclidean time manifold} \arrow[d, phantom, sloped, "="] & & \\
S^1 \arrow[r] & S^1 &
\end{tikzcd}$$

Path integral is an integral over the space of maps $S^1 \longrightarrow S^1$.

<u>General fact:</u> For all Gaussian integrals the saddle point approximation (physics: semiclassical approximation) is <u>EXACT</u>

First step: Find the saddle points – i.e find the solutions to eqs. of motion.

$$e^{-\frac{1}{\hbar}\int_0^\beta \frac{1}{2} I \dot{\phi}^2 dt - i\int_0^\beta \mathcal{B}\dot{\phi}\, dt}$$

$$\underline{\phi(0)=\phi_1 \, , \quad \phi(\beta)=\phi_2} \quad \longleftarrow$$

<u>Eqs:</u> $$\underline{\underline{\ddot{\phi}=0}} \quad \longleftarrow$$

<u>Solution:</u>

$$\phi(t) = \overset{\swarrow}{\phi_1} + \overset{\swarrow}{\frac{t}{\beta}}(\phi_2-\phi_1) \, , \quad 0\le t\le\beta$$

Not the only solution because

$\phi(t)\in\mathbb{R}$ rather $\phi(t)\in\mathbb{R}/2\pi\mathbb{Z}$!!!

$$e^{i\phi(t)} : S^1 \longrightarrow S^1$$

Can have <u>WINDING MODES</u>

The classical solutions are:

$$\boxed{\phi^{(w)}_{cl}(t) = \phi_1 + \frac{\phi_2 - \phi_1 + 2\pi w}{\beta}\, t}$$

$w \in \mathbb{Z}$

$$e^{i\phi^{(w)}_{cl}(t)} : \underset{\substack{\shortparallel \\ \text{Eucl.} \\ \text{time} \\ \text{circle}}}{S^1} \longrightarrow \underset{\substack{\shortparallel \\ \text{~~target~~} \\ \text{field space}}}{S^1}$$

Computer failure

winding # w.

$$\ddot{\phi} = 0 \qquad \begin{aligned} \phi(0) &= \phi_1 \mod 2\pi\mathbb{Z} \\ \phi(\beta) &= \phi_2 \mod 2\pi\mathbb{Z} \end{aligned}$$

In fact there are an ∞ # of solutions to classical equations of motion.

For historical reasons these are referred to as "instantons."

Going back to the path integral

$$\int [d\phi(t)]_{\phi(0)=\phi_1 \bmod 2\pi\mathbb{Z}}^{\phi(\beta)=\phi_2 \bmod 2\pi\mathbb{Z}} e^{-S_E}$$

In SP/SC analysis we expand around each stationary point and do the Gaussian integral around the stationary point and then sum over S.P. / Solutions of eqs. of motion. If $\phi_{cl}(t)$ is a solution:

$$\boxed{\phi(t) = \phi_{cl}(t) + \phi_q(t)}$$

$$S[\phi(t)] = S[\phi_{cl}(t)] + S[\phi_q(t)]$$

$$Z(\phi_2, \phi_1 | \beta) = Z_q \cdot \sum_{w \in \mathbb{Z}} e^{-S[\phi_{cl}(t)]}$$

$$= \sum_{w \in \mathbb{Z}} e^{-\frac{2\pi^2 I}{\beta}\left(w + \frac{\phi_2 - \phi_1}{2\pi}\right)^2} \cdot e^{2\pi i B\left(w + \frac{\phi_2 - \phi_1}{2\pi}\right)}$$

Z_q = path integral over ϕ_q

$$\phi(t) = \underline{\phi_{cl}^{(w)}(t)} + \phi_q(t)$$

$\phi_q(0) = \phi_q(\beta) = 0$ ~~$\bmod 2\pi\mathbb{Z}$~~

Regard $\phi_q \in \mathbb{R}$.

$$Z_q = \int [d\phi_q(t)]_{\phi_q(0)=0}^{\phi_q(\beta)=0} e^{-\int_0^\beta \frac{1}{2}\dot{\phi}^2 dt}$$

Gaussian Integrals

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x^2} = \sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-\frac{a}{2}x^2} = \frac{1}{\sqrt{a}}$$

$\mathrm{Re}(a) > 0$ for other values use analytic continuation.

A_{ij} symmetric matrix $1 \leq i,j \leq n$

$\mathrm{Re}(A_{ij}) > 0.$

$$\int \boxed{\prod_{i=1}^{n} \frac{dx^i}{\sqrt{2\pi}}} \exp\left(-\frac{1}{2} x^i A_{ij} x^j + b_i x^i\right)$$

$$= \frac{1}{\sqrt{\det A}} e^{\frac{1}{2} b_i (A^{-1})^{ij} b_j}$$

(∞ if A has a zero mode)

Generalize to ∞ - dimensions

$$\int [d\phi_q(t)] \exp - \int_0^1 \phi_q A \phi_q$$

$$A = -\frac{I}{2\beta} \frac{d^2}{d\tau^2} = \mathcal{O} \text{ on } L^2[0,1]$$

$A = \mathcal{O}$ has zeromode $\phi_q(t) = \text{cnst.}$

$\phi_q(0) = 0 = \phi_q(\beta)$.

$$Z_q = \frac{1}{\sqrt{\mathrm{Det}(\mathcal{O})}}$$

$$\mathcal{O} = -\frac{I}{2\beta}\frac{d^2}{d\tau^2}$$

Finite dimensions If A is diagonaliz.

then $\mathrm{Det}(A) = \prod_i \lambda_i$

Won't work in ∞ dimensions.

$\sin(n\tau)$ are eigenfunctions

$$\prod_{n=1}^{\infty}\left(\frac{I}{2\beta}n^2\right) \quad ??$$

Define the determinant using ζ-function regularization:

$$\frac{d}{ds}\Big|_{s=0} \lambda^{-s} = -\log\lambda \leftarrow$$

For $\mathcal{O}$ define a ζ-function

$$\zeta_{\mathcal{O}}(s) := \sum_{\lambda\neq 0} \lambda^{-s}$$

Formally

$$\operatorname{Det}\mathcal{O} = \exp\left(-\zeta_{\mathcal{O}}'(s)\Big|_{s=0}\right)$$

If $\zeta_{\mathcal{O}}(s)$ converges for $\mathrm{Re}(s)$ sufficiently large + positive and admits an a.c. to $s=0$

then we <u>DEFINE</u>

$$\mathrm{Det}\,\mathcal{O} := \exp\left(-\zeta_{\mathcal{O}}'(0)\right)$$

For $\mathcal{O} = -\frac{I}{2\hbar\beta}\frac{d^2}{d\tau^2}$

acting on $\mathbb{R}$-functions on $[0,1]$

w/ bc $\phi_q(0) = \phi_q(1) = 0$

$$\zeta_{\mathcal{O}}(s) = 2\left(\frac{I\pi^2}{2\hbar\beta}\right)^s \underset{\substack{\uparrow \\ \text{Riemann.}}}{\zeta(2s)}$$

$$\zeta(s) = -\frac{1}{2} + s\log\frac{1}{\sqrt{2\pi}} + O(s^2),$$

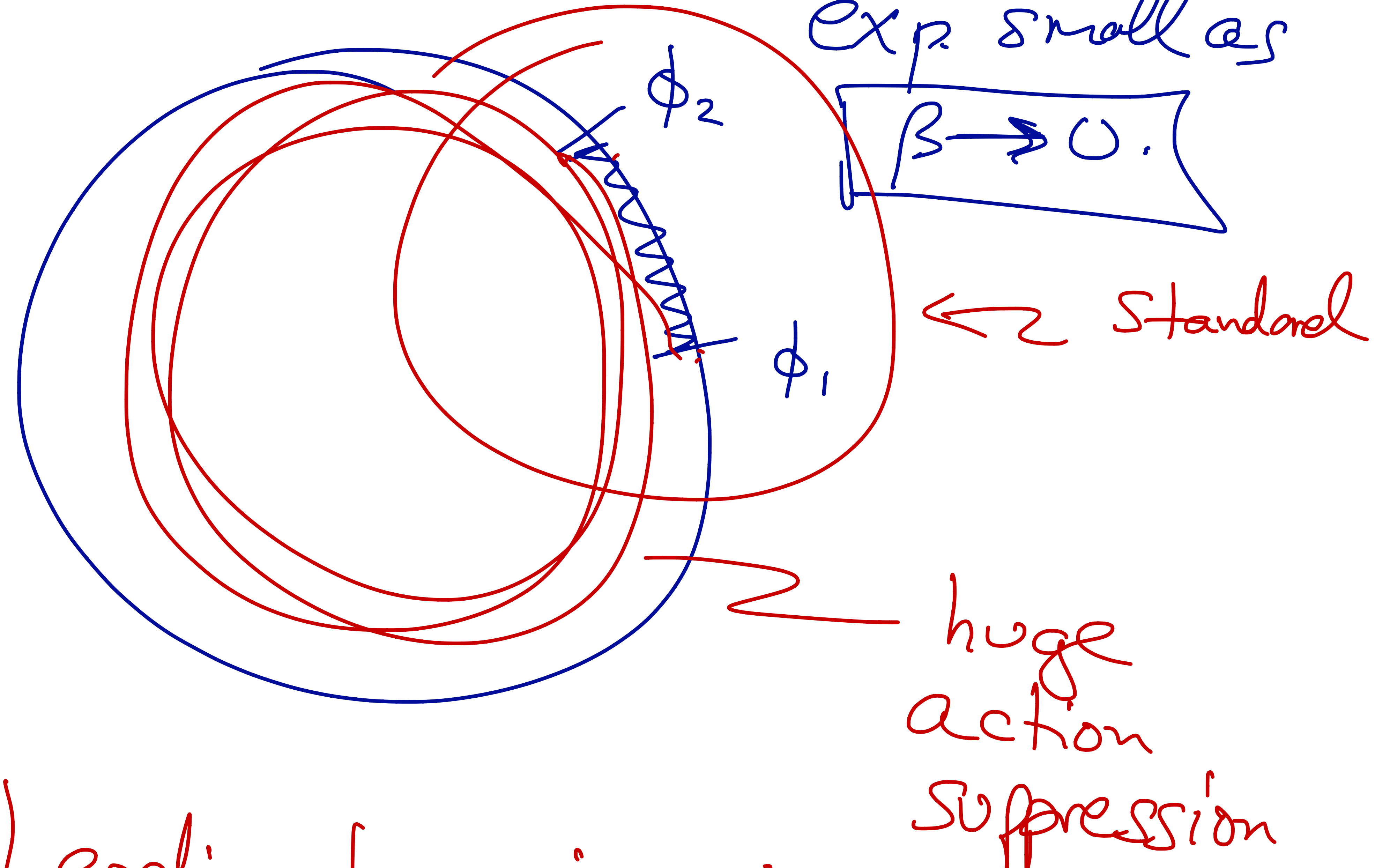
near $s=0$

$$\mathrm{Det}(\mathcal{O}) =' \frac{\beta}{I} \quad \text{up to } 2, \pi.$$

$\underline{\beta \longrightarrow 0}$

$$Z \to Z_q \, e^{-\frac{I}{2\beta}(\phi_2-\phi_1)^2 + i\theta(\phi_2-\phi_1)}$$

$$\left(1 + \underbrace{O\left(e^{-k/\beta}\right)}_{\text{exp small as } \boxed{\beta \to 0.}}\right)$$



Leading term is standard quantum propagator from $\phi_1 \to \phi_2$

$$Z_q = \sqrt{\frac{I}{\underline{\underline{2\pi\hbar}}\,\beta}}$$

Net result:

$$\underbrace{Z(\phi_2,\phi_1|\beta)} = \sqrt{\frac{I}{2\pi\hbar\beta}}$$

$$\circledast \sum_{\boxed{\omega\in\mathbb{Z}}} e^{-\frac{2\pi^2 I}{\beta}\left(\omega+\frac{\phi_2-\phi_1}{2\pi}\right)^2+2\pi i\vartheta\left(\omega+\frac{\phi_2-\phi_1}{2\pi}\right)}$$

path integ.

$\xleftarrow{\quad} \boxed{\beta\to 0}$

$$\underbrace{Z(\phi_2,\phi_1|\beta)} = \underbrace{\langle\phi_2|e^{-\beta H}|\phi_1\rangle} \checkmark$$

$$\to \quad = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{-\frac{\beta}{2I}(m-\vartheta)^2+im(\phi_2-\phi_1)} \longleftarrow$$

Hamilton viewpoint.

$$\uparrow$$

$$= \sum_m \langle\phi_2|\psi_m\rangle\langle\psi_m|e^{-\beta H}|\psi_m\rangle\langle\psi_m|\phi_1\rangle$$